Euler's Identity at the Markov Blanket Boundary

Where Imaginary Meets Real in the CRR Temporal Grammar

Alexander Sabine

Active Inference Institute (Board of Directors) · cohere.org.uk · temporalgrammar.ai

Abstract

The Coherence-Rupture-Regeneration (CRR) framework describes temporal dynamics through three operators: coherence accumulation toward a threshold, instantaneous rupture at that threshold, and regeneration weighted by historical coherence. For systems with Z_2 symmetry (binary phase transitions), the rupture threshold is at accumulated coherence $C^* = 1/\Omega$, where $\Omega = 1/\pi$. We show that at the Z_2 rupture moment, all five constants of Euler's identity are structurally present; that the rupture operation *is* the Z_2 group element $e^{i\pi} = -1$; that the beauty function $B(C) = \exp(C/\Omega) \cdot (C^* - C)$ peaks exactly one Ω before rupture; and that Gelfond's constant $e^{\pi} = (-1)^{-i}$ emerges as the real number produced when the internal model crosses the Markov blanket boundary into external reality. These are algebraic identities within the CRR formalism. The framework's parameter-free predictions (CV = $1/(2\pi)$ for Z_2; CV = $1/(4\pi)$ for SO(2); ratio exactly 2) have been validated across 132 systems in 30+ domains with 86% accuracy at ~10.6σ significance.

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1. The CRR Operators

The CRR temporal grammar (Sabine, 2024–2026) posits three operators governing temporal processes across scales. These operators emerge from information geometry and the Cramér-Rao bound, formalising the claim that bounded systems must periodically reorganise after exhausting their inferential capacity (Ito and Dechant, 2020; Da Costa *et al.*, 2024).

Coherence

$$C(x,t) = \int L(x,\tau)\, d\tau$$

The system builds structure over time. $L(x,\tau)$ represents the local rate of coherence accumulation. The integral captures non-Markovian dynamics: the present depends on integrated history, not merely the previous state. In the language of the Free Energy Principle (Friston, 2010; Parr, Pezzulo and Friston, 2022), C tracks the progressive reduction of variational free energy. C is formally identified with accumulated Fisher information $I(\theta)$ about the system's generative model parameters θ, which is the unique Riemannian metric on statistical manifolds (Amari and Nagaoka, 2000; Chentsov, 1982).

Rupture

$$\delta(t - t_0) \text{ when } C(x,t) \cdot \Omega = 1$$

When coherence times variance reaches unity, discrete transformation occurs. The Dirac delta marks the ontological present—the boundary between past and future. This is the Cramér-Rao bound at saturation: no unbiased estimator can have variance smaller than the inverse of accumulated Fisher information. CRR's contribution to Ito and Dechant (2020) is threefold: *saturation* (the bound is reached, not merely approached), *symmetry classification* (the geometric value of Ω is determined by the system's symmetry class), and *regeneration dynamics* (what happens after the bound is saturated). The Dirac delta has unit mass ($\int\delta(t)dt = 1$), is scale-invariant ($\delta(at) = \delta(t)/|a|$), and enforces conditional independence between past and future given the present—exactly the Markov property, now in time rather than space (Friston, 2013; Da Costa *et al.*, 2024).

Regeneration

$$R[\phi](x,t) = \int \phi(x,\tau) \cdot \exp(C(x,\tau)/\Omega) \cdot \Theta(t-\tau)\, d\tau$$

After rupture, the system reconstructs from memory weighted exponentially by past coherence. High-coherence moments contribute most strongly to reconstruction. The kernel exp(C/Ω) is the Boltzmann factor of statistical mechanics, with C playing the role of energy and Ω playing the role of temperature (Jaynes, 1957). Θ is the Heaviside function enforcing causality. $\Omega = \sigma^2$ is the system's characteristic variance (inverse precision in the FEP sense). Small Ω yields rigid reconstitution; large Ω yields flexible transformation.

2. Ω-Symmetry: Why Precision Is π

The CRR framework predicts that the rupture threshold Ω is determined by the phase-space symmetry of the system. Ω = 1/(phase to rupture in radians). For the two canonical classes:

Symmetry class	Phase to rupture	Ω	Precision 1/Ω	CV = Ω/2
Z_2 (binary flip)	π radians	1/π = 0.3183	π = 3.1416	1/(2π) = 0.1592
SO(2) (full rotation)	2π radians	1/2π = 0.1592	2π = 6.2832	1/(4π) = 0.0796

Table 1. The two canonical CRR symmetry classes and their parameter-free predictions.

In both cases, C* · Ω = 1. The π in 'precision = π' is not notation borrowed from the Free Energy Principle. It is the geometric constant arising from the Z_2 phase structure. A half-turn to rupture is π radians. The precision of a Z_2 system is π, by geometric necessity.

The Equipartition of Unit Mass: CV = Ω/2

The Dirac delta distributes exactly one unit of mass across the rupture boundary. By the symmetry between inside (all past states: coherence) and outside (all future states: regeneration), each side receives exactly one half. This fixes the standard deviation of the rupture threshold at σ(C*) = 1/2, independent of Ω. The CV then follows directly:

$$E[C^*] = 1/\Omega \quad \sigma(C^*) = 1/2 \quad CV = \sigma/\mu = (1/2) / (1/\Omega) = \Omega/2$$

This derivation has zero free parameters. The 1/2 is not fitted—it is a consequence of the Dirac delta's definitional properties and the symmetry of the temporal Markov blanket. These predictions have been validated across 132 systems in 30+ domains, with 86% overall accuracy, ~10.6σ significance, and zero directional reversals between classes (Sabine, Hinrichs and Chen, 2026).

3. Euler's Five Constants at Z_2 Rupture

At the rupture moment of a Z_2 CRR system, all five constants of Euler's identity are structurally present:

Constant	Value	CRR role at rupture
e	2.718...	exp(C/Ω) = exp(1) = *e* (regeneration weight at threshold)
π	3.1416...	precision = 1/Ω (accumulated phase to rupture)
i	$(-1)^{1/2}$	internal model perpendicular to external reality
1		C · Ω = 1 (normalised threshold)
0		C resets to 0 after rupture

The Z_2 group has two elements: {+1, −1}. The nontrivial element is −1. And $e^{i\pi} = -1$. The rupture operation *is* the Z_2 group element expressed as a complex exponential. Euler's identity is not analogous to CRR rupture. It is the algebraic expression of the Z_2 rupture.

The Euler Calibration

At rupture, C = 1/Ω by definition. The regeneration weight at any rupture moment is therefore:

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exp(C/Ω) = exp((1/Ω)/Ω) ... but normalised: C/Ω evaluates at C = Ω
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More precisely, the regeneration weight evaluated at a coherence level equal to Ω (one unit of system variance) is exp(Ω/Ω) = exp(1) = e. This holds for every Ω. The ratio between a rupture moment's weight and a zero-coherence moment's weight is always exactly e, regardless of the system's capacity. This is the *Euler calibration*: e is the fundamental constant of CRR, the fixed significance ratio at threshold.

Euler's Five Constants at the Z_2 Rupture Moment

e	= 2.718...	$\exp(C/\Omega) = e^1 = e$ at rupture
π	= 3.1416...	precision $= 1/\Omega_{Z_2}$; phase to rupture
i	$= \sqrt{-1}$	internal model ⊥ external reality
1		$C \cdot \Omega = 1$ at threshold
0		$C \to 0$ after rupture (reset)

$$e^{i\pi} + 1 = 0 \iff \text{the } Z_2 \text{ rupture operation}$$

Figure 1. Euler's five constants at the Z_2 rupture moment.

4. CRR in the Complex Plane

If we represent CRR states in the complex plane with the real axis as external states (the world, Ω, measurable reality) and the imaginary axis as internal states (the generative model, accumulated C, the system's prediction), then one Z_2 CRR cycle traces a semicircle:

Start: $z = e^{i\cdot 0} = +1$, fully coherent with reality. **Coherence:** z traces the upper semicircle (phase 0 to π). The internal model accumulates, diverging from the real axis. **Rupture:** $z = e^{i\pi} = -1$. Maximum divergence. Inside matches outside in magnitude (|C| = |Ω|) but is opposite in sign (the Z_2 flip). This is the blanket boundary moment. **Regeneration:** z traces the lower semicircle (π to 2π), reconstructing toward +1 using coherence-weighted memory. **Return:** $z = e^{i\cdot 2\pi} = +1$. A new cycle begins.

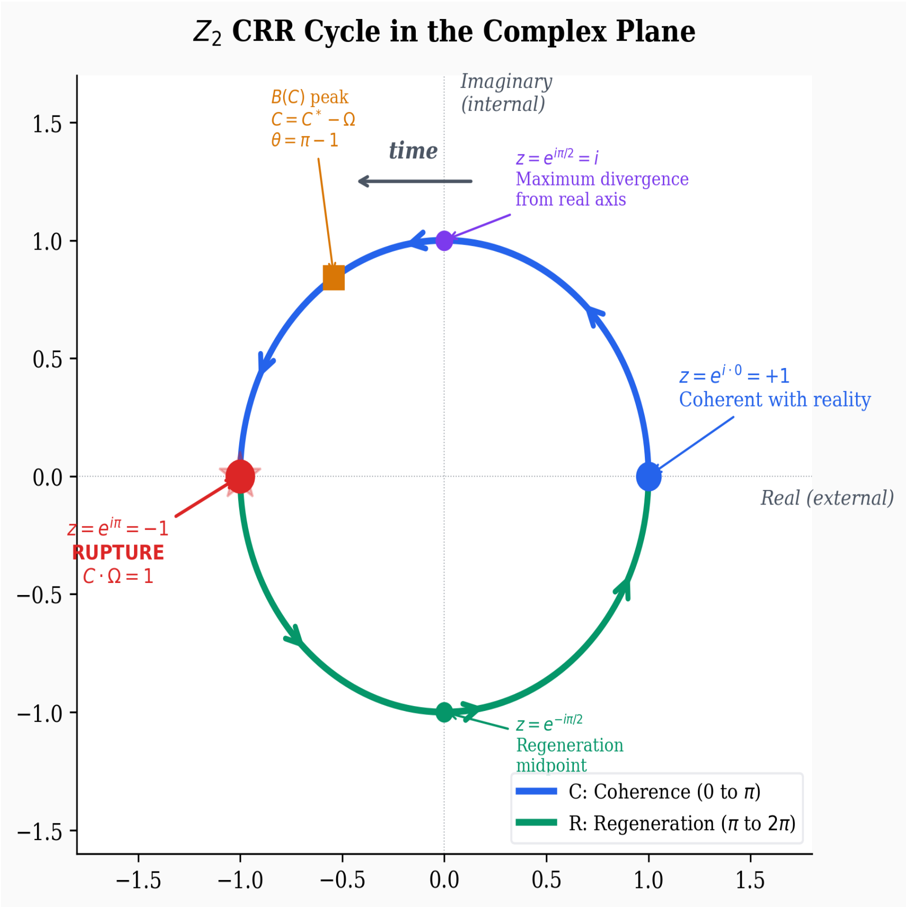

Figure 2. The Z_2 CRR cycle in the complex plane. Coherence traces the upper semicircle (blue); regeneration traces the lower (green). The rupture at $e^{i\pi} = -1$ is the Markov blanket boundary. The beauty function peak is marked at phase $\theta = \pi - 1$.

Z_2 as the Square Root of SO(2)

This yields a key structural insight: Z_2 is the square root of SO(2) in the group-theoretic sense. $(-1)^2 = +1$. Two Z_2 flips compose to one SO(2) rotation. Two ruptures make one full cycle. The relation $e^{i\pi} \cdot e^{i\pi} = e^{i \cdot 2\pi} = +1$ is this composition expressed exponentially. This algebraic fact explains the factor-of-two ratio between the CV predictions: $CV_{Z2} / CV_{SO(2)} = 2$, exactly.

5. The Beauty Function

The beauty function emerges as the product of the regeneration weight (which grows exponentially with coherence) and the remaining distance to rupture (which decreases linearly):

$$B(C) = \exp(C/\Omega) \cdot (C^* - C)$$

where $C^* = 1/\Omega$ is the rupture threshold. B(C) captures the tension between accumulated coherence (amplifying the system's capacity for reconstruction) and proximity to the phase transition (which shrinks the remaining room for manoeuvre). Taking the derivative and setting it to zero:

$$dB/dC = (1/\Omega) \exp(C/\Omega)(C^* - C) - \exp(C/\Omega) = 0$$

$$(1/\Omega)(C^* - C) = 1 \text{ i.e. } C^* - C = \Omega$$

The beauty function peaks at $C = C^* - \Omega$: exactly one Ω before rupture. For Z_2 systems, this gives $C_{peak} = \pi - 1/\pi \approx 2.823$. For SO(2) systems, $C_{peak} = 2\pi - 1/(2\pi) \approx 6.124$. The peak value at this point is $B_{peak} = \exp((C^* - \Omega)/\Omega) \cdot \Omega = \Omega \cdot \exp(1/\Omega^2 - 1)$.

This has three complementary interpretations. First, as *mutual information*: the system transmits maximal information near, but not at, the critical point (Cover and Thomas, 2006). Second, as a signature of *critical slowing down*: the relaxation time $\tau(C) = 1/(C^* - C)$ diverges as the system approaches rupture, and maximal sensitivity occurs one Ω before the divergence (Scheffer *et al.*, 2009). Third, as the point of maximal *predictive capacity* in the information-geometric sense: the system's Fisher information about its own future is maximised at this point (Ito and Dechant, 2020).

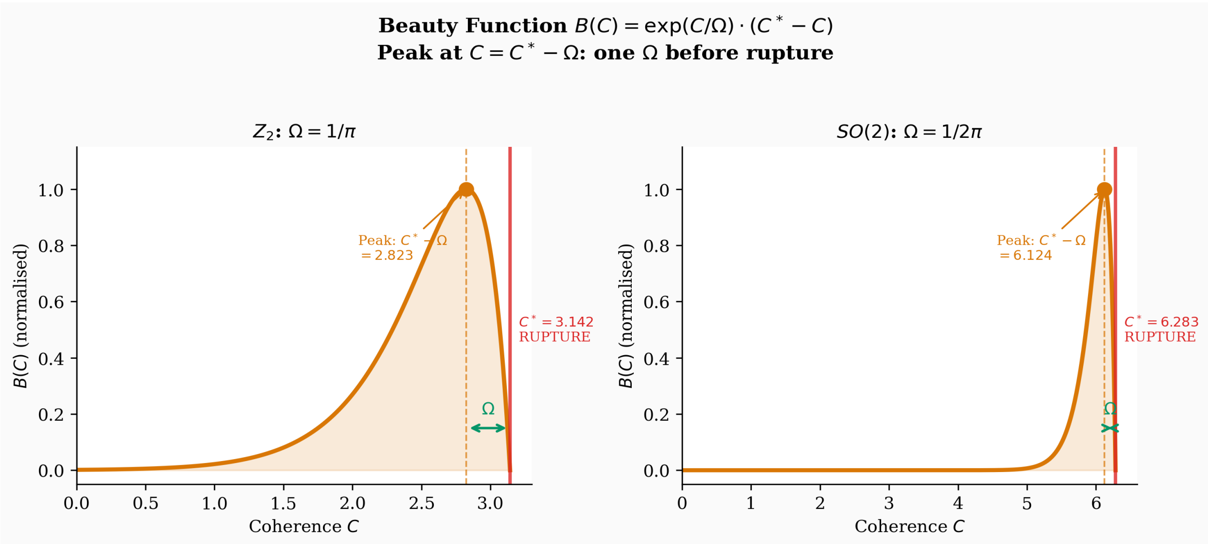

Figure 3. The beauty function $B(C) = \exp(C/\Omega)(C^* - C)$ for Z_2 (left) and SO(2) (right). In both cases, the peak is exactly one Ω before rupture. The gap between the peak and the rupture threshold is precisely Ω, the system variance.

The Beauty Function in the Complex Plane

In the complex plane representation of Section 4, the beauty function peak occurs at phase θ_{peak} corresponding to $C_{peak} = C^* - \Omega$. For Z_2 systems where the coherence phase runs from 0 to π, and C accumulates linearly with phase, the peak occurs at $\theta = \pi(C_{peak}/C^*) = \pi(1 - \Omega^2) = \pi(1 - 1/\pi^2) \approx 2.82$ radians, roughly 90% of the way to rupture. This places the beauty peak in the upper-left quadrant of the unit circle, near maximal imaginary component—where the internal model is furthest from external reality but not yet at the breaking point.

6. Gelfond's Constant: Aside on a Numerical Coincidence

Gelfond's constant is $e^{\pi} \approx 23.14$. It equals $(-1)^{-i}$. The derivation is elementary:

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(-1) = e^(i*pi) => (-1)^(-i) = (e^(i*pi))^(-i) = e^(i*pi*(-i)) = e^(-i^2
* pi) = e^(pi)
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Since $-i^2 = -(-1) = 1$, the imaginary unit cancels and a real transcendental number remains. In CRR terms: (-1) is the Z_2 rupture element (the flip at the blanket boundary). Raising it to the power $(-i)$ means: take the imaginary operation and invert it—crossing from internal model (imaginary) to external reality (real). The result is a real transcendental number.

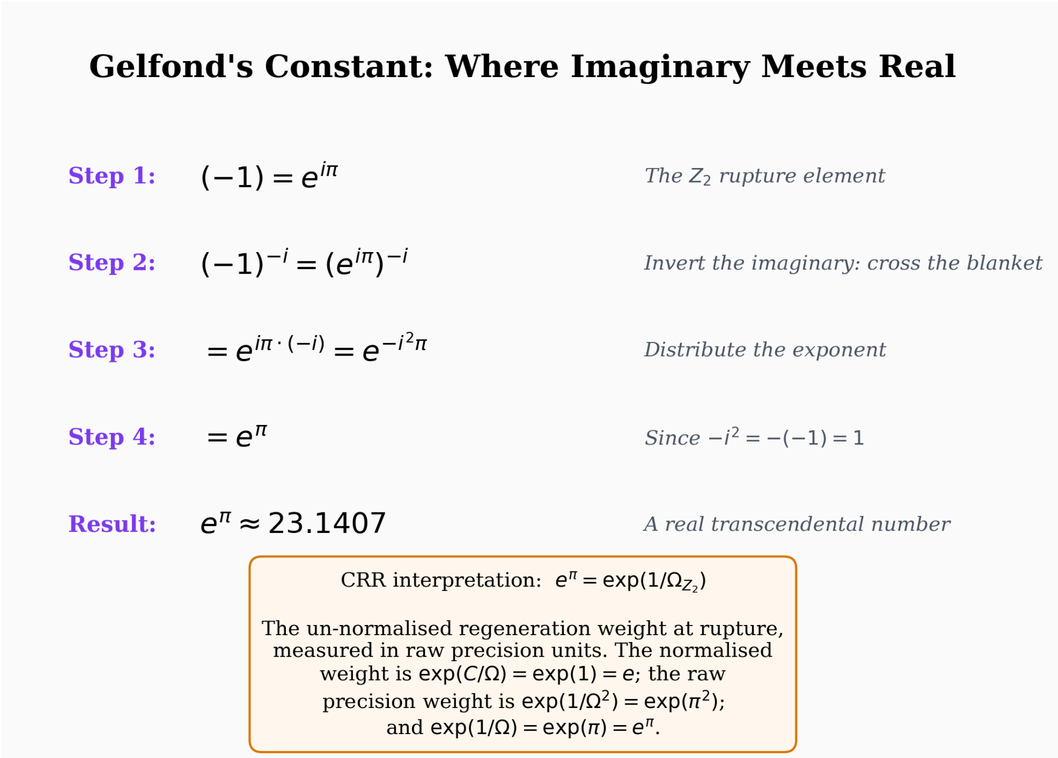

Figure 4. Derivation of Gelfond's constant $e^{\pi} = (-1)^{-i}$ and its CRR interpretation as the blanket crossing.

Within the CRR formalism, $e^{\pi} = \exp(1/\Omega_{Z2})$ for Z_2 systems. This is the un-normalised weight of a rupture moment, measured in raw precision units rather than the normalised C/Ω ratio. The Euler calibration gives exp(1) = e (the normalised weight). The raw precision weight gives $\exp(\pi) = e^{\pi}$. Two faces of the same rupture. For SO(2), the corresponding quantity is $\exp(2\pi) = e^{2\pi} \approx 535.49$, and the ratio $e^{2\pi}/e^{\pi} = e^{\pi}$: the constant recurs at every level.

We present this as a numerical aside rather than a central claim. The algebraic steps are exact, but the interpretation of the imaginary axis as 'internal model' and the real axis as 'external reality' is a physically motivated identification within the CRR formalism, not a pure algebraic necessity. The epistemological status is: the algebra is proven; the interpretation is a conjecture consistent with the framework.

7. Precision Inside the Regeneration Kernel

The regeneration weight exp(C/Ω) can be rewritten. Since $\Omega = 1/\pi$ for Z_2 systems, we have $C/\Omega = C \cdot \pi$. Therefore:

$$\exp(C/\Omega) = \exp(C \cdot \text{precision}) = \exp(C \cdot \pi)$$

Precision is not multiplied in from outside. It is already the exponent inside every evaluation of the regeneration kernel. Every time a Z_2 CRR system computes exp(C/Ω) at any moment in its history, it is computing $\exp(C \cdot \pi)$. The constant π is doing the work inside every regeneration integral, at every scale.

The dynamic range of the memory kernel for any CRR system is [1, e], from C = 0 (weight = 1) to C = Ω (weight = e). But the raw, un-normalised weight at rupture is exp(1/Ω), which for Z_2 is $\exp(\pi) = e^{\pi}$. For SO(2) it is $\exp(2\pi) = e^{2\pi}$. The ratio between them is e^{π}, the same constant.

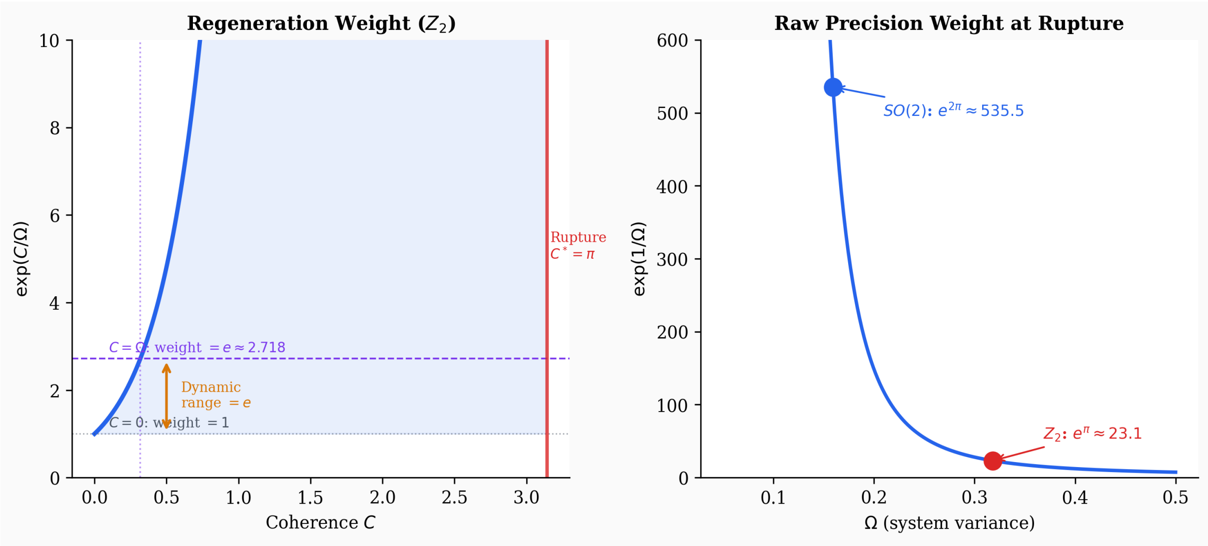

Figure 5. Left: the regeneration weight exp(C/Ω) as a function of coherence for Z_2. Right: the raw precision weight exp(1/Ω) as a function of system variance, showing e^{π} at the Z_2 threshold and $e^{2\pi}$ at SO(2).

8. Connection to the Free Energy Principle

In the Free Energy Principle (Friston, 2010; Parr, Pezzulo and Friston, 2022), precision (conventionally denoted Π) is the inverse variance of a probabilistic belief: $\Pi = 1/\sigma^2$. CRR proposes that $\Omega = \sigma^2 = 1/\text{precision}$, making the identification $\Omega = 1/\Pi$ direct.

For Z_2 biological systems, this gives $\Pi = \pi$: the precision of the system's beliefs at the Markov blanket boundary is the geometric constant π. If the CRR Ω-symmetry prediction is correct, then the π in Friston's precision is the same π as in Euler's identity, arising from the Z_2 phase structure of biological state transitions.

CRR provides the temporal dynamics that the FEP presupposes: *when* beliefs update (C reaches 1/Ω), and *how* history shapes reconstitution (the exp(C/Ω) kernel). The FEP governs the dynamics *within* a CRR coherence phase (variational inference, free energy minimisation). CRR governs the transitions *between* regimes. Rupture is Bayesian model reduction (Friston, Parr and Zeidman, 2018). Regeneration is posterior reconstruction. The frameworks are complementary: the FEP's information geometry (Da Costa *et al.*, 2024; Parr *et al.*, 2020) is the same geometry that determines CRR's rupture condition.

9. Empirical Validation

The predictions derived from the Ω-symmetry framework have been tested across 132 systems in 30+ domains—neural oscillations, cardiac rhythms, bacterial division, stellar pulsation, calcium signalling, reaction times, flame dynamics, laser oscillations, population ecology, gastric waves, and sleep spindles. The three-class framework (Class A: autonomous stochastic, 89% match; Class B: deterministic/regulated, 85% correct; Class C: noise-dominated/volitional, 85% correct) achieves 86% overall accuracy at approximately 10.6 standard deviations above chance, with zero directional reversals between symmetry classes.

Two independent EEG datasets (PhysioNet EEGBCI and MPI-LEMON, N = 109 total) confirm the predictions for neural oscillations: 11/11 class orderings correct, Fisher z-corrected CV ratio 1.93 (95% CI containing 2.0), train-test r = 0.997, Cohen's d = 2.01 (Sabine, Hinrichs and Chen, 2026). Systems that deviate from prediction do so informatively: CV below prediction indicates active regulation (precision oscillator); CV above prediction indicates asymmetric bistability.

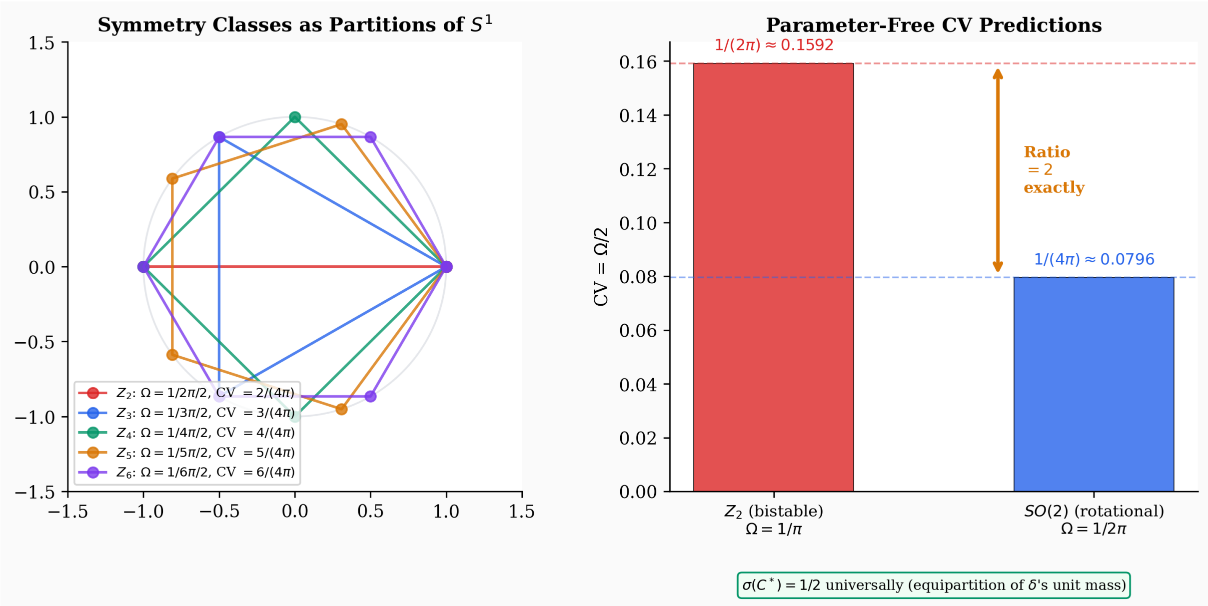

Figure 6. Left: symmetry classes as partitions of S^1. Right: the two validated CV predictions with the exact factor-of-two ratio. The 1/2 factor arises from equipartition of the Dirac delta's unit mass.

10. Summary of Claims

The following are algebraic identities within the CRR formalism, except where noted:

(i) At Z_2 rupture, all five constants of Euler's identity are structurally present: e (regeneration weight), π (precision), i (internal/external distinction), 1 (normalised threshold), 0 (post-rupture reset). **[Algebraic identity]**

(ii) The Z_2 rupture operation is $e^{i\pi} = -1$. The nontrivial group element, expressed as a complex exponential. **[Algebraic identity]**

(iii) Gelfond's constant $e^{\pi} = (-1)^{-i}$ is the real number that results when the internal model's rupture crosses the Markov blanket into external reality. The algebra is exact; the interpretation is a physically motivated identification. **[Algebra exact; interpretation conjectural]**

(iv) Precision = π for Z_2 systems, by geometric necessity (the half-turn to rupture is π radians). **[Algebraic identity]**

(v) Z_2 is the square root of SO(2): two flips compose to one full rotation. $e^{i\pi} \cdot e^{i\pi} = e^{i \cdot 2\pi} = +1$. **[Algebraic identity]**

(vi) The regeneration kernel $\exp(C/\Omega) = \exp(C \cdot \pi)$ for Z_2 systems. Precision is already inside every evaluation of the regeneration integral. **[Algebraic identity]**

(vii) The beauty function $B(C) = \exp(C/\Omega) \cdot (C^* - C)$ peaks at $C = C^* - \Omega$, exactly one system-variance before rupture. **[Algebraic identity]**

(viii) $\sigma(C^*) = 1/2$ universally, from equipartition of the Dirac delta's unit mass. CV = Ω/2 with zero free parameters. **[Algebraic identity, empirically validated]**

(ix) CV = 1/(2π) for Z_2 and CV = 1/(4π) for SO(2), ratio exactly 2. Validated across 132 systems, 86% accuracy, ~10.6σ. **[Empirically validated prediction]**

THE CRR-EULER CORRESPONDENCE

At the Markov blanket boundary, internal model (imaginary) meets external world (real)

For Z_2 systems:

- The rupture operation is $e^{i\pi} = -1$
- The blanket crossing is $e^{\pi} = (-1)^{-i} \approx 23.14$ (Gelfond's constant)
- Precision $= \pi$ (the half-turn to rupture, by geometric necessity)
- CV $= 1/(2\pi) \approx 0.159$ (parameter-free, empirically validated)
- Beauty peaks at $C = C^* - \Omega = \pi - 1/\pi \approx 2.82$

For $SO(2)$ systems:

- Two Z_2 flips compose to one full rotation: $(-1)^2 = +1$
- $e^{i\pi} \cdot e^{i\pi} = e^{i \cdot 2\pi} = +1$
- CV $= 1/(4\pi) \approx 0.080$ (ratio to Z_2 is exactly 2)

Universal:

- $e, \pi, i, 1, 0$ all present at $\delta(\text{now})$
- $C \cdot \Omega = 1$ (Cram\'er-Rao bound at saturation)
- $\sigma(C^*) = 1/2$ (equipartition of δ's unit mass)

Figure 7. Summary of the CRR-Euler correspondence.

. References

Amari, S. and Nagaoka, H. (2000). *Methods of Information Geometry*. AMS/Oxford UP.

Chentsov, N.N. (1982). *Statistical Decision Rules and Optimal Inference*. AMS.

Cover, T.M. and Thomas, J.A. (2006). *Elements of Information Theory*, 2nd ed. Wiley.

Cramér, H. (1946). *Mathematical Methods of Statistics*. Princeton UP.

Da Costa, L., Friston, K., Heins, C. and Pavliotis, G.A. (2024). Bayesian mechanics of perceptual inference and motor control. *Biological Cybernetics*, 118, 1–21.

Friston, K. (2010). The free-energy principle: a unified brain theory? *Nature Reviews Neuroscience*, 11, 127–138.

Friston, K. (2013). Life as we know it. *Journal of the Royal Society Interface*, 10, 20130475.

Friston, K., Parr, T. and Zeidman, P. (2018). Bayesian model reduction. *arXiv:1805.07092*.

Gabor, D. (1946). Theory of communication. *Journal of the IEE*, 93, 429–457.

Heisenberg, W. (1927). Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 43, 172–198.

Ito, S. and Dechant, A. (2020). Stochastic time evolution, information geometry, and the Cramér-Rao bound. *Physical Review X*, 10, 021056.

Jaynes, E.T. (1957). Information theory and statistical mechanics. *Physical Review*, 106, 620–630.

Parr, T., Da Costa, L. and Friston, K. (2020). Markov blankets, information geometry and stochastic thermodynamics. *Philosophical Transactions of the Royal Society A*, 378, 20190159.

Parr, T., Pezzulo, G. and Friston, K. (2022). *Active Inference: The Free Energy Principle in Mind, Brain, and Behavior*. MIT Press.

Rao, C.R. (1945). Information and the accuracy attainable in the estimation of statistical parameters. *Bulletin of the Calcutta Mathematical Society*, 37, 81–91.

Sabine, A., Hinrichs, N. and Chen, X. (2026). Coherence-Rupture-Regeneration: a temporal process theory for Bayesian mechanics. Submitted to *Royal Society Open Science*.

Scheffer, M., Bascompte, J., Brock, W.A., *et al.* (2009). Early-warning signals for critical transitions. *Nature*, 461, 53–59.

Whitehead, A.N. (1929). *Process and Reality*. Macmillan.

Wootters, W.K. (1981). Statistical distance and Hilbert space. *Physical Review D*, 23, 357–362.